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POLES OF L-FUNCTIONS, PERIODS AND UNITARY THETA. PART I: ODD TO EVEN

BY

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ABSTRACT

In this paper we prove the equivalence between the non-vanishing of the Θ correspondence on an irreducible, generic, cuspidal representation of U_{2n+1}, the non-vanishing of a certain generalized period and the existence of a pole of a twisted partial L-function.

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1. Introduction

We try in this paper to relate the poles of the partial L-function of a cuspidal generic representation π of the quasi-split unitary group $\mathrm{U}_{n+1,n}$ to the vanishing or non-vanishing of its lift under the theta correspondence to $\mathrm{U}_{n,n}$. These properties are already known to be connected in several situations; let us cite $(\mathrm{Sp}(2n), \mathrm{O}(2n))$ [GRS3] and $(\mathrm{U}_2, \mathrm{U}_3)$ [GRS1]. In all cases, including ours, the

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connection between the two properties is the non-vanishing of a certain generalized period integral.

To begin with, let us introduce the main notations of this article. We fix a number field F and E a quadratic extension of F, and denote their respective adele rings by $\mathbf{A}_F$ and $\mathbf{A}_E$. We will denote either by $c(x)$ or by $\bar{x}$ the action of the non-trivial element of the Galois group of E over F on an element x of E. Let w_n be the antidiagonal matrix of size n, that is the element

$$w_n = \begin{pmatrix} 0 & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & 0 \end{pmatrix}$$

of $\mathrm{GL}(n, \mathbf{Z})$. The group U_n will be the algebraic subgroup of $\mathrm{GL}(n)$ defined over F by

$$\mathrm{U}_n = \{u \in \mathrm{GL}(n) \| {}^{\mathrm{t}}\overline{u} w_n u = w_n\}.$$

We let G_n be U_{2n+1}, and H_l, $l > 0$ the algebraic subgroup of $\mathrm{GL}(2l)$ defined by

$$H_l = \left\{ u \in \mathrm{GL}(2l) \| {}^{\mathrm{t}}\overline{u} \begin{pmatrix} & w_l \\ -w_l & \end{pmatrix} u = \begin{pmatrix} & w_l \\ -w_l & \end{pmatrix} \right\}.$$

Note that H_l is non-canonically isomorphic to U_{2l}. When g is an element of $\mathrm{GL}(n, \mathbf{A}_E)$, we let g^* be the element of $\mathrm{GL}(n, \mathbf{A}_E)$ such that

$$\begin{pmatrix} g & & \\ & 1 & \\ & & g^* \end{pmatrix} \in G_n(\mathbf{A}_F),$$

namely $g^* = w_n {}^{\mathrm{t}}\overline{g}^{-1} w_n$. We define N_n to be the unipotent subgroup of upper triangular matrices of $\mathrm{GL}(n)$ with unit diagonal. We also let X be the subgroup of upper triangular matrices with unit diagonal in G_n.

Let π be an irreducible automorphic cuspidal representation of $G_n(\mathbf{A}_F)$, acting in a given space of cusp forms, which we keep denoting by π. We fix once and for all a non-trivial character $\psi_\circ$ of $\mathbf{A}_F$ trivial on F and we consider $\psi = \psi_\circ \circ \mathrm{Tr}_{E/F}$. For any element φ of the space of π let

$$W_\varphi(g) = W_\varphi^\psi(g) = \int_{X(F)\backslash X(\mathbf{A}_F)} \varphi(ng)\psi\Big(\sum n_{i,i+1}\Big)^{-1} \mathrm{d}n$$

be its Whittaker–Fourier coefficient with respect to ψ. The space of all W_φ^ψ is the ψ-Whittaker model $\mathcal{W}(\pi, \psi)$ of π. We assume that this space is non-zero, i.e., π is (globally) generic with respect to ψ. We want to introduce the generalized period we will deal with. Consider the subgroup U of G_n whose elements are such that the middle 3×3 block is I_3. For a matrix u in $U(\mathbf{A}_F)$, define $\psi_U(u) = \psi(\sum_{i=1}^{n-2} u_{i,i+1} + u_{n-1,n+1})$. Define for $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{U}_2$,

$$i(g) = \begin{pmatrix} I_{n-1} & & & & \\ & a & & b & \\ & & 1 & & \\ & c & & d & \\ & & & & I_{n-1} \end{pmatrix}.$$

Then $i(\mathrm{U}_2)$ acts by conjugation on U and preserves the character ψ_U. For any function φ in π and any character μ of $\mathbf{A}_E^\times$, we define

$$P_\psi(\varphi, \mu) = \int_{\mathrm{U}_2(F)\backslash \mathrm{U}_2(\mathbf{A}_F)} \int_{U(F)\backslash U(\mathbf{A}_F)} \varphi(ui(g))\psi_U(u)^{-1}\mu(\det g)dudg.$$

Let S_0 be the (finite) set of places v of F such that either $v|\infty$, or E/F, or μ ramifies at one of the places of E above v, or π or ψ ramifies at v.

The main result is the following:

THEOREM 1.1: *The following conditions are equivalent.*

1. *The partial* L*-function* $\mathrm{L}^S(\pi \times \mu, s)$ *has a (necessarily simple) pole at* $s = 1$ *for any finite set of prime* $S \supset S_0$.
2. *The generalized period integral* $P_\psi(., \mu)$ *does not vanish identically on* π.
3. *The representation* $\pi \otimes \mu \circ \det$ *has a* Θ *lifting to a generic cuspidal representation of* H_n *for some choice of splitting data.*

The paper is organized as follows. In section 2, we describe the L-function machine based on an integral similar to a Rankin–Selberg integral, with an additional integration along a unipotent subgroup. This integration will lead, when we take a residue at $s = 1$, to the generalized period P_ψ. In section 3, we prove the implication $1 \Rightarrow 2 \Rightarrow 3$ and some result on the vanishing of the Θ correspondence. In section 4, we prove that $3 \Rightarrow 1$ by showing that $\mathrm{L}^S(\pi \times \mu, s)$ is the product of a zeta function by a non-vanishing partial L-function.

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2. Product decomposition

Let us begin by decomposing the integral we will use to represent the L-function.

2.1. Fourier Series. We will need to use the decomposition of a certain constant term in Fourier series. Let U_0 be the subgroup of G_n of the form

$$\begin{pmatrix} I_n & & a \\ & 1 & \\ & & I_n \end{pmatrix}.$$

Proposition 2.1: *For any φ in the space of π and any g in $G_n(\mathbf{A}_F)$, we have:*

$$\int_{U_0(F)\backslash U_0(\mathbf{A}_F)} \varphi(ug)\mathrm{d}u = \sum_{\gamma\in \mathrm{N}_n(E)\backslash \mathrm{GL}(n,E)} W_\varphi^\psi(\begin{pmatrix} \gamma & & \\ & 1 & \\ & & \gamma^* \end{pmatrix} g).$$

Proof: This is a consequence of formula (1.0.1), p. 59 of [GPSR]. Let S_n be the subgroup of G_n comprising matrices of the form

$$\begin{pmatrix} A & v & X \\ & 1 & v' \\ & & A^* \end{pmatrix}$$

with $A \in \mathrm{GL}(n)$, v any vector and X any matrix that fits. Then U_0 is a normal subgroup of S_n and we denote P_{n+1} the quotient $U_0\backslash S_n$. We observe that P_{n+1} is isomorphic to the subgroup $\mathrm{Res}_{E/F} P_{n+1}$ of [GPSR, p. 58]. The unipotent radical of $P_{n+1}(F)$ is isomorphic to $\mathrm{N}_{n+1}(E)$; the quotient is isomorphic to $\mathrm{N}_n(E)\backslash \mathrm{GL}(n, E)$ where $\mathrm{GL}(n, E)$ is injected in S_n by

$$\gamma \longmapsto \begin{pmatrix} \gamma & & \\ & 1 & \\ & & \gamma^* \end{pmatrix}.$$

In addition, if for $p \in S_n$ we define

$$f(p) = \int_{U_0(F)\backslash U_0(\mathbf{A}_F)} \varphi(upg)\mathrm{d}u,$$

then f defines a cuspidal function $\tilde{f}$ on $P_{n+1}(F)\backslash P_{n+1}(\mathbf{A}_F)$. We have

$$\int_{\mathrm{N}_{n+1}(E)\backslash \mathrm{N}_{n+1}(\mathbf{A}_E)} \tilde{f}(n)\psi_{\mathrm{N}_{n+1}}(n)^{-1}\mathrm{d}n = W_\varphi(g),$$

where $\mathrm{N}_{n+1}(\mathbf{A}_E)$ is injected in $P_{n+1}(\mathbf{A}_F)$ by the obvious morphism ($n \in \mathrm{N}_{n+1}(\mathbf{A}_E)$ is sent to the class of any element of $S_n(\mathbf{A}_F)$ that has n in its upper-left-hand $(n+1) \times (n+1)$ corner). The result of the proposition is then a consequence of abovementioned formula (1.0.1) applied to $\tilde{f}(1) = f(1) = \int_{U_0(F)\backslash U_0(\mathbf{A}_F)} \varphi(ug)\mathrm{d}u$. ∎

2.2. The Integral.

Definition 2.1: Let μ be a character of $\mathbf{A}_E^\times$ trivial on $E^\times$. Let B_2 be the subgroup of upper triangular matrices in U_2. An element of B_2 will be of the form $\begin{pmatrix} z & x \\ & \bar{z}^{-1} \end{pmatrix}$. For any $s \in \mathbf{C}$, we consider the character μ_s of $B_2(\mathbf{A}_F)$ whose value on the aforementioned matrix is $\mu(z)|z|^{s-1/2}$. We define $I_{\mu,s}$ to be the space of holomorphic K-finite functions in the space of the unitarily induced representation of μ_s from $B_2(\mathbf{A}_F)$ to $\mathrm{U}_2(\mathbf{A}_F)$. Then, for any $f_s \in I_{\mu,s}$, $E^{\mathrm{U}_2}(f_s, g) = \sum f_s(\gamma g)$ for γ running over a set of representatives of $\mathrm{U}_2(F)$ mod $B_2(F)$ (on the left).

Let $i = i_{2n+1}$ be the injection of U_2 in G_n defined by

$$i_{2n+1}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} I_{n-1} & & & & \\ & a & & b & \\ & & 1 & & \\ & c & & d & \\ & & & & I_{n-1} \end{pmatrix}.$$

We recall that U is the subgroup of G_n of matrices of the form

$$u = \begin{pmatrix} z & x & a \\ & I_3 & x' \\ & & z^* \end{pmatrix}$$

with $z \in \mathrm{N}_{n-1}$. We define the character ψ_U by

$$\psi_U\colon U(\mathbf{A}_F) \to \mathbf{C}^\times$$
$$u \mapsto \psi\Big(\sum z_{i,i+1} + x_{n-1,2}\Big)$$

with u as above. We can then define, for $\varphi \in \pi$,

$$\varphi^\psi(g) = \int_{U(F)\backslash U(\mathbf{A}_F)} \varphi(ug)\psi_U(u)^{-1}\mathrm{d}u.$$

Definition 2.3: For φ an element of π, μ a character of $\mathbf{A}_E^\times/E^\times$, $s \in \mathbf{C}$ and $f_s \in I_{\mu,s}$, define

$$I(\varphi, f_s) = \int_{\mathrm{U}_2(F)\backslash \mathrm{U}_2(\mathbf{A}_F)} \varphi^\psi(i(g)) E^{\mathrm{U}_2}(f_s, g)\mathrm{d}g.$$

We set R to be the algebraic subgroup of G_n of matrices of the form

$$\begin{pmatrix} 1 & & & & \\ y & I_{n-1} & & & \\ & & 1 & & \\ & & & I_{n-1} & \\ & & & y' & 1 \end{pmatrix}$$

(with y any $n-1$-dimensional vector). For W an element of $\mathcal{W}(\pi, \psi)$ and the other elements as above, define

$$Z(W, f_s) = \int_{N_2(\mathbf{A}_F)\backslash \mathrm{U}_2(\mathbf{A}_F)} \int_{R(\mathbf{A}_F)} W(rw_0 i(g)) f_s(g)\mathrm{d}r\mathrm{d}g$$

with

$$w_0 = \begin{pmatrix} & 1 & & & \\ I_{n-1} & & & & \\ & & 1 & & \\ & & & & I_{n-1} \\ & & & 1 & \end{pmatrix}$$

and N_2 the unipotent radical of B_2.

This integral will provide the L-function we study. It is the Rankin–Selberg convolution for $G_n \times \mathrm{GL}(1)$ and can be generalized for any $\mathrm{GL}(k)$, $k \leq n$.

PROPOSITION 2.4: *For φ, $s \in \mathbf{C}$ with real part large enough and f_s as above,*

$$I(\varphi, f_s) = Z(W_\varphi^\psi, f_s).$$

Proof: We begin by unfolding the Eisenstein series. This is possible as long as we take the real part of s large enough. One has

$$\begin{aligned} I(\varphi, f_s) &= \int_{\mathrm{U}_2(F)\backslash \mathrm{U}_2(\mathbf{A}_F)} \varphi^\psi(i(g)) \sum_{\gamma \in B_2(F)\backslash \mathrm{U}_2(F)} f_s(\gamma g)\mathrm{d}g \\ &= \int_{\mathrm{U}_2(F)\backslash \mathrm{U}_2(\mathbf{A}_F)} \sum_{\gamma \in B_2(F)\backslash \mathrm{U}_2(F)} \varphi^\psi(i(g)) f_s(\gamma g)\mathrm{d}g \\ &= \int_{\mathrm{U}_2(F)\backslash \mathrm{U}_2(\mathbf{A}_F)} \sum_{\gamma \in B_2(F)\backslash \mathrm{U}_2(F)} \varphi^\psi(i(\gamma g)) f_s(\gamma g)\mathrm{d}g \end{aligned}$$

where this last equality holds because φ is invariant by multiplication on the left by $i(\gamma)$ and when

$$u = \begin{pmatrix} z & x & a \\ & I_3 & x' \\ & & z^* \end{pmatrix},$$

$i(\gamma)ui(\gamma)^{-1}$ is the same matrix with x replaced by $xi_3(\gamma)^{-1}$, so that the central column of x is unchanged and so the character will not be affected. The integral will thus remain the same after a change of variable. We then collapse the sum with the integral:

$$I(\varphi, f_s) = \int_{B_2(F)\backslash \mathrm{U}_2(\mathbf{A}_F)} \varphi^{\psi}(i(g))f_s(g)\mathrm{d}g.$$

Let T_2 be the diagonal torus of U_2.

$$\begin{aligned} I(\varphi, f_s) &= \int_{T_2(F)N_2(\mathbf{A}_F)\backslash \mathrm{U}_2(\mathbf{A}_F)} \int_{N_2(F)\backslash N_2(\mathbf{A}_F)} \varphi^{\psi}(i(n)i(g))f_s(ng)\mathrm{d}n\mathrm{d}g \\ &= \int_{T_2(F)N_2(\mathbf{A}_F)\backslash \mathrm{U}_2(\mathbf{A}_F)} \int_{N_2(F)\backslash N_2(\mathbf{A}_F)} \varphi^{\psi}(i(n)i(g))\mathrm{d}nf_s(g)\mathrm{d}g. \end{aligned}$$

We now consider changing the inner integral. For any $g \in G_n(\mathbf{A}_F)$, let

$$\begin{aligned} I' &= \int_{N_2(F)\backslash N_2(\mathbf{A}_F)} \varphi^{\psi}(i(n)g)\mathrm{d}n \\ &= \int_{U'(F)\backslash U'(\mathbf{A}_F)} \varphi(ug)\psi_{U'}(u)^{-1}\mathrm{d}u \end{aligned}$$

with

$$U' = \{u \in G_n || u = \begin{pmatrix} z & x & a \\ & 1 & x' \\ & & z^* \end{pmatrix}, z \in \mathrm{N}_n, x_n = 0\}$$

and

$$\psi_{U'}(u) = \psi\Big(\sum_{i=1}^{n-2} z_{i,i+1} + x_{n-1}\Big).$$

We note that there are two differences with the usual notation: the character is not taken on the last column of z and is taken on the last but one component of x, which is the last non-zero component. Decomposing the subgroup U',

$$I' = \int_{\mathrm{N}_{n-1}(E)\backslash \mathrm{N}_{n-1}(\mathbf{A}_E)} \int_{\mathbf{A}_E^{n-1}/E^{n-1}} \int_{\mathbf{A}_E^{n-1}/E^{n-1}} \int_{U_0(F)\backslash U_0(\mathbf{A}_F)}$$
$$\varphi\left(u\begin{pmatrix} z & y & x & & * \\ & 1 & 0 & & \\ & & 1 & 0 & x' \\ & & & 1 & y' \\ & & & & z^* \end{pmatrix} g\right)\psi\Big(\sum z_{i,i+1} + x_{n-1}\Big)^{-1} \mathrm{d}u\mathrm{d}x\mathrm{d}y\mathrm{d}z.$$

We found it convenient, though maybe less natural, to introduce w_0 immediately and to conjugate it from left to right. We can introduce it on the left of the argument of φ and conjugate all the matrices up to g (excluded). What remains is

$$\varphi\left(u\begin{pmatrix} 1 & 0 & 0 & & * \\ y & z & x & & \\ & & 1 & x' & 0 \\ & & & z^* & 0 \\ & & & y' & 1 \end{pmatrix} w_0 g\right)$$

with a change of variable in the a part of u (the last column becomes the first and the last row becomes the first and all other rows and columns shift in the natural way). Using Proposition 2.1,

$$I' = \int_{\mathbf{A}_E^{n-1}/E^{n-1}} \int_{\mathrm{N}_{n-1}(E)\backslash \mathrm{N}_{n-1}(\mathbf{A}_E)} \int_{\mathbf{A}_E^{n-1}/E^{n-1}} \sum_{\gamma\in \mathrm{N}_n(E)\backslash \mathrm{GL}(n,E)}$$
$$W_\varphi^\psi\left(\begin{pmatrix} \gamma & & \\ & 1 & \\ & & \gamma^* \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & & * \\ y & z & x & & \\ & & 1 & x' & 0 \\ & & & z^* & 0 \\ & & & y' & 1 \end{pmatrix} w_0 g\right)$$
$$\psi\Big(\sum z_{i,i+1} + x_{n-1}\Big)^{-1} \mathrm{d}x\mathrm{d}z\mathrm{d}y.$$

We want to remark here on the order of integration: it is important to choose the right order to preserve the invariance of the partial integral under some rational subgroups.

The idea is that we integrate a character over a compact group in the two innermost integrals and this character must be trivial. This poses conditions on the form of γ. Namely, the integral over x tells one that the last row of γ must be of the form $(a_n, 0, \ldots, 0, 1)$; the first coefficient is not determined because the first element of the column containing x is 0. This is valid since the series on

$\gamma \in \mathrm{N}_n(E)\backslash \mathrm{GL}(n,E)$ is square summable. Then, since we sum modulo $\mathrm{N}_n(E)$, we might as well suppose that the last column of γ is ${}^t(0,\dots,0,1)$. We find that γ is of the form $\begin{pmatrix} \gamma' & 0 \\ a_n \quad 0 & 1 \end{pmatrix}$. We now have $\gamma' \in \mathrm{GL}(n-1,E)$ and an easy induction shows that a set of representatives of γ in the sum can be chosen of the form

$$\begin{pmatrix} \gamma & \\ a & I_{n-1} \end{pmatrix}$$

with $\gamma \in E^*$ and $a \in E^{n-1}$. If we factor out the matrix $\begin{pmatrix} \gamma & \\ & I_{n-1} \end{pmatrix}$ we find

$$\begin{aligned} I' &= \sum_{\gamma\in E^*}\sum_{a\in E^{n-1}} \int_{R(F)\backslash R(\mathbf{A}_F)} W_\varphi^\psi\left(\begin{pmatrix} \gamma & & \\ & I_{2n-1} & \\ & & \overline{\gamma}^{-1}\end{pmatrix} \begin{pmatrix} 1 & & & & \\ a & I_{n-1} & & & \\ & & 1 & & \\ & & & I_{n-1} & \\ & & & a' & 1 \end{pmatrix} r w_0 g\right) \mathrm{d}r \\ &= \sum_{\gamma\in E^*}\sum_{a\in E^{n-1}} \int_{\mathbf{A}_E^{n-1}/E^{n-1}} W_\varphi^\psi\left(\begin{pmatrix} \gamma & & \\ & I_{2n-1} & \\ & & \overline{\gamma}^{-1}\end{pmatrix} \begin{pmatrix} 1 & & & & \\ y+a & I_{n-1} & & & \\ & & 1 & & \\ & & & I_{n-1} & \\ & & & y'+a' & 1 \end{pmatrix} w_0 g\right) \mathrm{d}y \\ &= \sum_{\gamma\in E^*} \int_{R(\mathbf{A}_F)} W_\varphi^\psi\left(\begin{pmatrix} \gamma & & \\ & I_{2n-1} & \\ & & \overline{\gamma}^{-1}\end{pmatrix} r w_0 g\right) \mathrm{d}r \\ &= \sum_{\gamma\in E^*} \int_{R(\mathbf{A}_F)} W_\varphi^\psi\left(r\begin{pmatrix} \gamma & & \\ & I_{2n-1} & \\ & & \overline{\gamma}^{-1}\end{pmatrix} w_0 g\right) \mathrm{d}r. \end{aligned}$$

We arrive at

$$I(\varphi, f_s) = \int_{T_2(F)N_2(\mathbf{A}_F)\backslash \mathrm{U}_2(\mathbf{A}_F)} \sum_{\gamma\in E^*} \int_{R(\mathbf{A}_F)} W_\varphi^\psi\left(r\begin{pmatrix} \gamma & & \\ & I_{2n-1} & \\ & & \overline{\gamma}^{-1}\end{pmatrix} w_0 i(g)\right) \mathrm{d}r f_s(g)\mathrm{d}g$$

$$= \int_{T_2(F)N_2(\mathbf{A}_F)\backslash \mathrm{U}_2(\mathbf{A}_F)} \sum_{\gamma \in E^*}$$

$$\int_{R(\mathbf{A}_F)} W_\varphi^\psi \left(r w_0 \begin{pmatrix} I_{n-1} & & & & \\ & \gamma & & & \\ & & 1 & & \\ & & & \overline{\gamma}^{-1} & \\ & & & & I_{n-1} \end{pmatrix} i(g) \right) \mathrm{d}r f_s(g) \mathrm{d}g$$

$$= \int_{T_2(F)N_2(\mathbf{A}_F)\backslash \mathrm{U}_2(\mathbf{A}_F)} \sum_{\gamma \in E^*}$$

$$\int_{R(\mathbf{A}_F)} W_\varphi^\psi \left(r w_0 i(\begin{pmatrix} \gamma & \\ & \overline{\gamma}^{-1} \end{pmatrix} g) \right) \mathrm{d}r f_s(g) \mathrm{d}g.$$

We can put the matrix containing γ into f_s as well since μ is trivial on E^* and collapse the sum and the integral. ∎

Remark 2.5: As suggested by the referee, a different proof is possible along the lines of [Gin].

2.3. Non-ramified Computations.

2.3.1 Split places. Let v be a place of F such that E/F splits at v and not in S_0. Let r be an integer. The group $\mathrm{U}_r(F_v)$ is isomorphic to $\mathrm{GL}_r(F_v)$. To see that isomorphism concretely, let us denote by w one of the places of E above v; then the other one will be wc. The non-trivial element of $\mathrm{Gal}(E/F)$ sends E_w to E_{wc}; actually, these two fields are extensions of F_v of degree 1. The action of c is just the isomorphism induced by the equality $E_w = F_v = E_{wc}$ (as extensions of F_v). An element of $\mathrm{U}_r(F_v)$ is a pair (g_1, g_2) of $\mathrm{GL}_r(E_w) \times \mathrm{GL}_r(E_{wc})$ such that $g_2 = w_r c({}^{\mathrm{t}}g_1{}^{-1}) w_r$. We will then identify $\mathrm{U}_r(F_v)$ with $\mathrm{GL}_r(F_v)$ by projection onto the first component and identifying E_w and F_v. Let us see precisely what this means for $B_2(F_v)$. An element of $B_2(F_v)$ will be a pair of elements of $\mathrm{GL}_r(F_v)$

$$\left(\begin{pmatrix} a_1 & b_1 \\ & d_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ & d_2 \end{pmatrix} \right)$$

such that $d_2 = a_1^{-1}$, $a_2 = d_1^{-1}$ and $b_2 = -a_1^{-1} d_1^{-1} b_1$. When we take the character $\mu(z)$ of an element $\begin{pmatrix} z & x \\ & \overline{z}^{-1} \end{pmatrix}$, this amounts to $\mu_w(a_1)\mu_{wc}(a_2) = \mu_w(a_1)\mu_{wc}(d_1^{-1})$ in these coordinates.

Proposition 2.6: *Let W° be the essential vector (in the sense of [JPSS, définition (4.4), p. 211]) of the Whittaker model of π_v. Let f_s be the element of*

$I_{\mu,s,v}$ *which is identically 1 on* $\mathrm{GL}_2(\mathcal{O}_v)$*. Then*

$$Z(W^\circ, f_s) = \frac{\mathrm{L}(\pi_v \times \mu_w, s)\mathrm{L}(\check{\pi}_v \times \mu_{wc}, s)}{\mathrm{L}(\mu_w\mu_{wc}, 2s)}.$$

Proof: We can first conjugate w_0 to the right of the argument of W°. By doing so, the elements of GL_2 move to the top, left, bottom and right of GL_{2n+1}. We can then cancel w_0, as well as the integration along the maximal compact of GL_2. What remains is

$$Z(W^\circ, f_s) = \int_{(F_v^\times)^2}\int_{(F_v{}^{n-1})^2} W^\circ(\begin{pmatrix} a & & & & \\ y & I_{n-1} & & & \\ & & 1 & & \\ & & & I_{n-1} & \\ & & & -{}^{\mathrm{t}}z & d^{-1} \end{pmatrix})\mathrm{d}y\mathrm{d}z$$
$$\mu_w(a)\mu_{wc}(d)|ad|^{s-1/2}|ad|^{-1/2}|a|^{1-n}|ad|^{-1}\mathrm{d}^\times a\mathrm{d}^\times d$$

(the $|ad|^{-1/2}$ comes from the unitary induction, the $|a|^{1-n}$ from the Haar measure for the y part and the $|ad|^{-1}$ from the Haar measure on U_2). The $-{}^{\mathrm{t}}z$ and d^{-1} come from the particular isomorphism we chose between $G_n(F_v)$ and $\mathrm{GL}_{2n+1}(F_v)$. To evaluate this integral, we will have to compute the values of the function W°. It is explicitly computed in [JS1, section 2] provided we have a decomposition of its argument in Bruhat form nak. For unramified data, the k part can be ignored and the n as well, as soon as it is in $\mathrm{N}_{2n+1}(\mathcal{O}_v)$.

First we claim that we can do the Bruhat decomposition in each GL_n separately and reunite them blockwise. Then, we can see that the y integration can be ignored: a Bruhat decomposition of the matrix $\left(\begin{smallmatrix} a & \\ y & I_{n-1}\end{smallmatrix}\right)$ is

$$\begin{pmatrix} a & \\ & \mathrm{diag}(a_1, \dots, a_{n-1}) \end{pmatrix}\begin{pmatrix} 1 & & & & \\ y_1/a_1 & a_1^{-1} & & & \\ y_2/a_2 & & a_2^{-1} & & \\ & \dots & & & \\ y_{n-1}/a_{n-1} & & & & a_{n-1}^{-1} \end{pmatrix}$$

with $a_i = \gcd(y_i, 1)$. As soon as $y_i \notin \mathcal{O}_v$, a_i is a negative power of the uniformizer. This means that the a part of the argument of W° will have increasing powers of the uniformizer somewhere (because there is 1 in the center) and then W° is zero. So we must have all y_i in $\mathcal{O}_v$ and then we have a clear Bruhat decomposition

$$\begin{pmatrix} a & \\ & I_{n-1} \end{pmatrix}\begin{pmatrix} 1 & \\ y & I_{n-1} \end{pmatrix},$$

so the value of W° does not depend on $y \in {\mathcal{O}_v}^{n-1}$ and thus the integration gives

$$W^\circ \begin{pmatrix} a & & & & \\ & I_{n-1} & & & \\ & & 1 & & \\ & & & I_{n-1} & \\ & & & -{}^{t}z & d^{-1} \end{pmatrix}.$$

It is also obvious that a must be in $F_v^\times \cap \mathcal{O}_v$. We claim that we must have $d \in F_v^\times \cap \mathcal{O}_v$ and $z \in d^{-1}{\mathcal{O}_v}^{n-1}$ and that the integral in z is just multiplication by $|d|^{1-n}$. What remains is

$$\begin{aligned} Z(W^\circ, f_s) &= \int_{({\mathcal{O}_v}^\times)^2} W^\circ \begin{pmatrix} a & & \\ & I_{2n-1} & \\ & & d^{-1} \end{pmatrix} \mu_w(a)\mu_{wc}(d)|ad|^s|ad|^{-n}\mathrm{d}^\times a\mathrm{d}^\times d \\ &= \sum_{a,d\in\mathbf{N}} W^\circ \begin{pmatrix} \varpi^a & & \\ & I_{2n-1} & \\ & & \varpi^{-d} \end{pmatrix} \mu_w(\varpi)^a\mu_{wc}(\varpi)^d q^{(a+d)(n-s)} \end{aligned}$$

with ϖ a uniformizer of F_v and q the number of elements of its residual field.

To simplify notations, we let $\mu = \mu_w$, $\mu' = \mu_{wc}$ and $\pi = \pi_v$. We will compare the formal series in q^{-s} given by $\mathrm{L}(\mu\mu', 2s)Z(W^\circ, f_s)$ and $\mathrm{L}(\pi\times\mu, s)\mathrm{L}(\tilde{\pi}\times\mu', s)$. We want first to recall the value of the class one Whittaker function from [JS1, section 2]. Let r be an integer and $\mathcal{W}$ a class one Whittaker function of $\mathrm{GL}_r(F_v)$ such that $\mathcal{W}(I_r) = 1$. Then if $J = (j_1, \ldots, j_r) \in \mathbf{Z}^r$, ϖ^J will be $\mathrm{diag}(\varpi^{j_i})$ and $\mathcal{W}(\varpi^J) = 0$ unless $j_1 \geq j_2 \geq \cdots \geq j_r$, in which case $\mathcal{W}(\varpi^J) = \delta(\varpi^J)^{1/2}\operatorname{Tr}(\rho_J(A))$ with δ the module of B_r, ρ_J is the highest weight module with highest weight J and A is the Satake parameter of the unramified representation generated by $\mathcal{W}$. This gives us the expression for the integral $Z(W^\circ, f_s)$:

$$\begin{aligned} Z(W^\circ, f_s) &= \sum_{a,d\in\mathbf{N}} \operatorname{Tr}(\rho_{(a,0,\ldots,0,-d)}(A))\mu(\varpi)^a\mu'(\varpi)^d q^{-(a+d)s} \\ &= \sum_{k\in\mathbf{N}}\sum_{a=0}^{k} \operatorname{Tr}(\rho_{(a,0,\ldots,0,a-k)}(A))\mu(\varpi)^a\mu'(\varpi)^{k-a}q^{-ks}. \end{aligned}$$

Next, recall the expression for the local L factor $\mathrm{L}(\pi\times\mu, s)$, given in [JS2, section 1]:

$$\mathrm{L}(\pi\times\mu, s) = \sum_{a\in\mathbf{N}} \operatorname{Tr}(\rho_{(a,0,\ldots,0)}(A))\mu(\varpi)^a q^{-as}$$

with, as before, A the Satake parameter of π. Now, if $\tilde{\pi}$ is the contragredient of π, its Satake parameter is A^{-1}. But then $\operatorname{Tr}(\rho_J(A^{-1})) = \operatorname{Tr}(\rho_{\tilde{J}}(A))$, with

$\check{J} = (-j_n, \ldots, -j_1)$ the "contragredient" weight of the dominant weight $J = (j_1, \ldots, j_n)$. We get

$$\mathrm{L}(\check{\pi} \times \mu', s) = \sum_{a \in \mathbf{N}} \mathrm{Tr}(\rho_{(0,\ldots,0,-a)}(A))\mu'(\varpi)^a q^{-as}.$$

The last remark is that

$$\mathrm{Tr}(\rho_J(A))\,\mathrm{Tr}(\rho_{J'}(A)) = \mathrm{Tr}((\rho_J \otimes \rho_{J'})(A)).$$

So,

$$\begin{aligned} &\mathrm{L}(\pi \times \mu, s)\mathrm{L}(\check{\pi} \times \mu', s) \\ &= \sum_{k \in \mathbf{N}} \sum_{a=0}^{k} \mathrm{Tr}(\rho_{(a,0,\ldots,0)}(A))\,\mathrm{Tr}(\rho_{(0,\ldots,0,a-k)}(A))\mu(\varpi)^a \mu'(\varpi)^{k-a} q^{-ks} \\ &= \sum_{k \in \mathbf{N}} \sum_{a=0}^{k} \mathrm{Tr}((\rho_{(a,0,\ldots,0)} \otimes \rho_{(0,\ldots,0,a-k)})(A))\mu(\varpi)^a \mu'(\varpi)^{k-a} q^{-ks}. \end{aligned}$$

The two expressions for $Z(W^\circ, f_s)$ and the product of L-factors are very similar. The only difference is that the first has

$$\rho_{(a,0,\ldots,0,a-k)}$$

while the other one has

$$\rho_{(a,0,\ldots,0)} \otimes \rho_{(0,\ldots,0,a-k)}.$$

But we know how highest weight modules tensorize. Since these two are very simple (they are just symmetric powers), the decomposition is very easy: it is the direct sum of all the $\rho_{(b,0,\ldots,0,2a-k-b)}$ with $0 \le b \le a$. What this means is that when a varies, we will get all the $\rho_{(b,\ldots,-c)}$ with $b+c$ of the same parity as k and $b - c \le k$; each of these will be obtained only once. This looks like all the terms of $Z(W^\circ, f_s)$ of degree at most k, each multiplied by the appropriate power of $\mu(\varpi)\mu'(\varpi)q^{-2s}$. Thus

$$\mathrm{L}(\pi \times \mu, s)\mathrm{L}(\check{\pi} \times \mu', s) = \left(\sum_{k \in \mathbf{N}} (\mu(\varpi)\mu'(\varpi)q^{-2s})^k\right) Z(W^\circ, f_s).$$

The sum is exactly the local L-factor $\mathrm{L}(\mu\mu', 2s)$. ∎

2.3.2 Inert places. The computation is very similar. It is actually easier because the variables do not split. Let v be a place of F such that E/F is inert at v and $v \notin S_0$. We will denote w the place of E above v. Let W° be the spherical element of the Whittaker model of π_v with value 1 at I_{2n+1}. The local integral we have to compute is

$$Z(W^\circ, f_s) = \int_{E_w^\times} \int_{E_w^{n-1}} W^\circ \begin{pmatrix} a & & & & \\ y & I_{n-1} & & & \\ & & 1 & & \\ & & & I_{n-1} & \\ & & & y' & a^* \end{pmatrix} \mathrm{d}y$$
$$\mu_w(a)|a|^{s-1/2}|a|^{1/2}|a|^{1-n}|a|^{-1}\mathrm{d}^\times a$$

with, as before, the $|a|^{1/2}$ coming from the unitary induction, the $|a|^{1-n}$ from the (right invariant) Haar measure on the parabolic subgroup and the $|a|^{-1}$ from the Haar measure on U_2. As before y must have integral coefficients (note that then the coefficients of y' will be divisible by a^{-1} as in the split case) and a must be in $\mathcal{O}_v \cap F_v^\times$. Since y varies in a dimension-1 space, integration over the y variable cancels. We arrive at something very similar to what we had in the split case:

$$\begin{aligned} Z(W^\circ, f_s) &= \sum_{a \in \mathbf{N}} W^\circ \begin{pmatrix} \varpi^a & & \\ & I_{2n-1} & \\ & & \varpi^{-a} \end{pmatrix} \mu_w(\varpi)^a q^{-as} \\ &= \sum_{a \in \mathbf{N}} \mathrm{Tr}(\rho^{\mathrm{Sp}_{2n}}_{(a,0,\ldots,0)})(A)\mu_w(\varpi)^a q^{-as} \end{aligned}$$

with A the Satake parameter of π_v and $\rho_\lambda^{\mathrm{Sp}_{2n}}$ the irreducible highest weight module of highest weight λ of $\mathrm{Sp}_{2n}(\mathbf{C})$. This sum is in turn equal to the local L-function $\mathrm{L}(\pi_v \times \mu_w, s)$.

PROPOSITION 2.7: *Let W° be the essential vector for the Whittaker model of π_v. Let f_s be the element of $I_{\mu,s,v}$ equal to 1 on $\mathrm{U}_2(\mathcal{O}_v)$. Then*

$$Z(W^\circ, f_s) = \mathrm{L}(\pi_v \times \mu_w, s).$$

2.4. SOME NON-VANISHING RESULTS. We will show that for any $s_0 \in \mathbf{C}$, for any place we can always choose local data so that the local integral is non-vanishing for $s = s_0$.

PROPOSITION 2.8: *Let v be a place of F and s_0 a complex number. There exist a W in the Whittaker model of π_v and $f_s \in I_{\mu,s,v}$ such that*

$$Z(W, f_s)! = 0.$$

The proof will occupy the rest of this section. There are three non-archimedean cases, depending on the behaviour of each place in the extension and only one archimedean case.

2.4.1 Split places. Let v be a non-archimedean place where E/F is split. The local integral is equal to

$$Z(W,f_s)=\int_{N_2(F_v)\backslash \mathrm{GL}(2,F_v)}\int_{({F_v}^{n-1})^2}W(\begin{pmatrix}1&&&&\\ y&I_{n-1}&&&\\ &&1&&\\ &&&I_{n-1}&\\ &&&-{}^{\mathrm t}z&1\end{pmatrix}w_0 i(g))\,\mathrm{d}y\mathrm{d}z f_s(g)\mathrm{d}g.$$

The Whittaker function W is right invariant under the action of a compact open subgroup K_W of $\mathrm{GL}(n,F_v)$. We choose f_s to be equal to 1 on $i^{-1}(K_W)$ (which is a compact open subgroup of $\mathrm{GL}(2,F_v)$) and 0 on its complementary in $\mathrm{GL}(2,\mathcal{O}_v)$. Then

$$Z(W,f_s)=c\int_{(F_v^\times)^2}\int_{({F_v}^{n-1})^2}W(\begin{pmatrix}a&&&&\\ y&I_{n-1}&&&\\ &&1&&\\ &&&I_{n-1}&\\ &&&-d^{-1}\cdot{}^{\mathrm t}z&d^{-1}\end{pmatrix}w_0)\mathrm{d}y\mathrm{d}z$$
$$\mu_w(a)\mu_{wc}(d)|ad|^{1/2-s}|ad|^{1/2}|a|^{1-n}\mathrm{d}^\times a\mathrm{d}^\times d$$

with $c>0$ being the measure of $i^{-1}(K_W)$.

We will then eliminate one by one the components of y and z as follows. For some pair of integers k, k', let

$$W_1(g)=\int_{|t|\le q^k,|u|\le q^{k'}}W(gw_0(I_{2n+1}+tE_{1,n+1}-uE_{n+1,2n+1})w_0)\mathrm{d}t\mathrm{d}u.$$

Then

$$Z(W_1,f_s)=\int_{(F_v^\times)^2}\int_{({F_v}^{n-1})^2}\int_{|t|\le q^k,|u|\le q^{k'}}$$
$$W(\begin{pmatrix}a&&&&\\ y&I_{n-1}&&&\\ &&1&&\\ &&&I_{n-1}&\\ &&&-d^{-1}{}^{\mathrm t}z&d^{-1}\end{pmatrix}w_0$$
$$(I_{2n+1}+tE_{1,n+1}-uE_{n+1,2n+1}))\mathrm{d}t\mathrm{d}u\mathrm{d}y\mathrm{d}z$$
$$\mu_w(a)\mu_{wc}(d)|ad|^{2-n-s}\mathrm{d}^\times a\mathrm{d}^\times d$$

$$= \int_{(F_v^\times)^2} \int_{({F_v}^{n-2})^2} \int_{|t|\leq q^k, |u|\leq q^{k'}} \psi(ty_{n-1} + uz_{n-1}) \mathrm{d}t \mathrm{d}u$$

$$W(\begin{pmatrix} a & & & & \\ y & I_{n-2} & & & \\ & & I_3 & & \\ & & & I_{n-2} & \\ & & & -d^{-1\mathrm{t}}z & d^{-1} \end{pmatrix} w_0) \mathrm{d}y \mathrm{d}z$$

$$\mu_w(a)\mu_{wc}(d)|ad|^{2-n-s} \mathrm{d}^\times a \mathrm{d}^\times d.$$

We may now choose k and k' large enough so that the inner integral is zero unless

$$I_{2n+1} + y_{n-1}E_{n-1,1} - z_{n-1}E_{2n+1,n+2} \in K_W.$$

We then have

$$Z(W_1, f_s) = c_1 \int_{(F_v^\times)^2} \int_{({F_v}^{n-2})^2} W(\begin{pmatrix} a & & & & \\ y & I_{n-2} & & & \\ & & I_3 & & \\ & & & I_{n-2} & \\ & & & -d^{-1\mathrm{t}}z & d^{-1} \end{pmatrix} w_0) \mathrm{d}y \mathrm{d}z$$

$$\mu_w(a)\mu_{wc}(d)|ad|^{2-n-s} \mathrm{d}^\times a \mathrm{d}^\times d.$$

We go on by induction until we arrive at

$$Z(W_{n-1}, f_s) = c_{n-1} \int_{(F_v^\times)^2} W(\begin{pmatrix} a & & \\ & I_{2n-1} & \\ & & d^{-1} \end{pmatrix} w_0)$$

$$\mu_w(a)\mu_{wc}(d)|ad|^{2-n-s} \mathrm{d}^\times a \mathrm{d}^\times d.$$

With

$$\tilde{W}(g) = \int_{|t|\leq q^k, |u|\leq q^{k'}} W(gw_0(I_{2n+1} + tE_{1,2} - uE_{2n,2n+1})w_0)\psi(t+u)^{-1} \mathrm{d}t \mathrm{d}u,$$

we have similarly

$$Z(\tilde{W}_{n-1}, f_s) = cW(w_0)$$

for some non-zero constant c. It is then easy to check that one can find in the Whittaker model of π_v a Whittaker function which is non-zero on w_0.

2.4.2 Inert places. Let v be a non-archimedean place where E/F is inert. The computation is entirely similar to the split case; one has to replace d by $\bar{a}$ ($|ad|$ by $|a|$), z by $\bar{y}$ and u by $\bar{t}$.

2.4.3 Ramified places. If v is a place where E/F ramifies, then there is no extension of local fields so that $E_w = F_v$ and the computation is exactly the same as above once we take into account that $\bar{x} = x$ for all $x \in E_w$.

2.4.4 Archimedean places. We want to prove that the local integral does not vanish. We will bring the problem to the usual L-function problem with the following lemma.

LEMMA 2.9: *Let π_v be a generic irreducible representation of $G_n(F_v)$ with Whittaker model $\mathcal{W} = \mathcal{W}(\pi_v, \psi_v)$ and $s \in \mathbf{C}$. The integral $Z(W, f_s)$ is non-vanishing on $\mathcal{W} \times I_{\mu,s,v}$ if and only if*

$$Z'(W, f_s) = \int_{N_2(\mathbf{A}_F)\backslash U_2(\mathbf{A}_F)} W(w_0 i(g)) f_s(g) \mathrm{d}g$$

does not vanish on the same space.

Proof: We see that the integrals are very similar; we just have to eliminate the variables in R. This will be done recursively using the Dixmier–Malliavin lemma. We know from [DM] that any W in $\mathcal{W}$ can be written as a linear combination of functions of the form

$$g \longmapsto \int_{F_v} \Phi(x) W_1(g(I_{2n+1} + xE_{1,n} - \bar{x}E_{2n+1,n+2}))\mathrm{d}x$$

with $\Phi \in \mathcal{S}(F_v)$. This leads to

$$Z(W, f_s) = \sum_{W_1} \int_{N_2(\mathbf{A}_F)\backslash U_2(\mathbf{A}_F)} \int_{R_1(F_v)} W_1(rw_0 i(g)) f_s(g) \hat{\Phi}(r_{n-1})\mathrm{d}r$$

with

$$R_1 = \left\{ \begin{pmatrix} 1 & & & & \\ r_1 & & & & \\ \vdots & I_{n-1} & & & \\ r_{n-1} & & & & \\ & & 1 & & \\ & & & I_{n-1} & \\ & & & * & 1 \end{pmatrix} \right\}.$$

Since Φ, and thus $\hat{\Phi}$, is arbitrary in $\mathcal{S}(F)$, the integral will not vanish if and only if

$$\int_{N_2(\mathbf{A}_F)\backslash U_2(\mathbf{A}_F)} \int_{R_1(F_v)} W_1(rw_0 i(g)) f_s(g)\mathrm{d}r$$

does not vanish for some W_1. This proves the lemma for $n = 1$. The induction step is easy and follows the same lines, replacing R_i by R_{i+1} where for $i \leq n$

$$R_i = \left\{ \begin{pmatrix} 1 & & & & \\ r_1 & & & & \\ \vdots & I_{n-i} & & & \\ r_{n-i} & & & & \\ & & I_{2i-1} & & \\ & & & I_{n-i} & \\ & & & * & 1 \end{pmatrix} \right\}.$$ ∎

The proof of the non-vanishing of the local factor is based on the following lemma.

LEMMA 2.10: *Let π_v be a representation of $G_n(F_v)$ that is irreducible and generic. Let f_s be a family of elements of $I_{\mu,s,v}$ with the same restriction (as s varies) to K_2. The local integral $Z'(W, f_s)$ is convergent for real part of s large and can be continued meromorphically in s to $\mathbf{C}$. Moreover, the meromorphic continuation is continuous in each of its arguments.*

Proof: This is a consequence of the asymptotic expansion of the Whittaker functions. The proof follows the lines of the proof for any such integral. ∎

Using the lemma, we can prove

PROPOSITION 2.11: *For any $s_0 \in \mathbf{C}$ we can find elements W and f_s such that the local integral $Z(W, f_{s_0})$ is non-zero.*

Proof: We proceed with $Z'(W, f_s)$ since this is equivalent. Assume that $Z'(W, f_s)$ is 0 at $s = s_0$ for all choices of data. Let K_2 be the maximal compact subgroup of $\mathrm{U}_2(\mathbf{R})$. For real part of s large, we have, thanks to Iwasawa decomposition,

$$Z'(W, f_s) = \int_{K_2} Z_1(W, s, k) f_s(k) \mathrm{d}k$$

with

$$Z_1(W, s, k) = \int_{\mathbf{R}^*} W\left(w_0 i \begin{pmatrix} a & \\ & a^* \end{pmatrix} i(k)\right) |a|^{s-n} \mathrm{d}^\times a.$$

Since f_s can be chosen as we wish on $B_2 \cap K_2 \backslash K_2$, it follows that Z_1 is zero for any choice of the data W and k. Thus, with $k = I_2$, and for any W,

$$\int_{\mathbf{R}^*} W\left(w_0 i \begin{pmatrix} a & \\ & a^* \end{pmatrix}\right) |a|^{s_0-n} \mathrm{d}^\times a = 0.$$

Now, if we replace W by

$$\int W_1(g(I_{2n+1}+vE_{12}-\bar{v}E_{2n,2n+1}))\Phi(v)\mathrm{d}v,$$

we get that for any W, $W(w_0)=0$, which is false. ∎

3. Theta correspondence

3.1. Generalized Period. Let us suppose that $\mathrm{L}^S(\pi\times\mu,s)$ has a pole at $s=1$ for any finite set of places S such that $S\supset S_0$. Let $W=\otimes_v' W_v$ be a (pure tensor) element of $\mathcal{W}(\pi,\psi)$. According to 2.3, for any s such that there is no pole and any $f_s\in I_{\mu,s}$, $\mathrm{L}(\pi\times\mu,s)$ and $Z(W,f_s)$ are equal up to a finite set S of places (including the places at infinity). Increase S such that $S_0\subset S$. According to 2.4, for any $v\in S$ we can choose a $W_v^*\in\mathcal{W}(\pi_v,\psi_v)$ and a cross-section $f_{s,v}^*\in I_{\mu,s,v}$ such that $Z(W_v^*,f_{s,v}^*)$ is non-zero at $s=1$. We change W and f_s such that their local component at v is resp. W_v^* and $f_{s,v}^*$ for $v\in S$. The pole of the L-function must come from the Eisenstein series on U_2 and thus is simple, with non-zero residue. The residue of the Eisenstein series is $\mu\circ\det$. This means that the integral

$$P_\psi(\varphi,\mu)=\int_{\mathrm{U}_2(F)\backslash\mathrm{U}_2(\mathbf{A}_F)}\int_{U(F)\backslash U(\mathbf{A}_F)}\varphi(ui(g))\psi_U(u)^{-1}\mu(\det g)\mathrm{d}u\mathrm{d}g$$

is non-zero for φ the element corresponding to W. It is this period that will provide the link between the pole of the partial L-function and the theta correspondence.

3.2. Setup. We set up the data needed for the discussion of the image of the Howe lift between G_n and the tower of H_l, $l>0$.

The group G_n will act on the right of the vectors while H_l will act on the left. Let v be a finite place of F such that E/F is inert at v. Let τ be an admissible, irreducible and generic representation of $G_n=G_n(F_v)$ and for l integer, σ an admissible, irreducible representation of $H_l=H_l(F_v)$ which pairs with τ. This means that there is a $G_n\times H_l$ equivariant map T:

$$T\colon\omega_\psi^{(n,l)}\otimes\sigma\to\tau,$$

where $\omega_\psi^{(n,l)}$ is the Weil representation of $\widetilde{\mathrm{Sp}}(4l(2n+1))$ restricted to the dual pair $G_n\times H_l$. We will denote the space of Schwartz–Bruhat functions on an F_v-vector space V as $\mathcal{S}(V)$.

Note that the space on which H_l acts is split. Let $(W, \langle .,. \rangle)$ be the space on which G_n acts; we choose a basis $(e_1, \dots, e_n, e_0, e_{-n}, \dots, e_{-1})$ with respect to which the form has matrix

$$\begin{pmatrix} & & w_n \\ & 1 & \\ w_n & & \end{pmatrix}.$$

Put $W^{\pm} = \mathrm{Vect}(e_{\pm i})_{1 \le i \le n}$. Similarly, let $(V, (.,.))$ be the space on which H_l acts, let $(f_1, \dots, f_l, f_{-l}, \dots, f_{-1})$ be a basis of V and let $V^{\pm} = \mathrm{Vect}(f_{\pm j})_{1 \le j \le l}$. We thus have $W = W^+ \oplus (e_0) \oplus W^-$ and $V = V^+ \oplus V^-$; for any vector $v \in V$, we write $v = v^+ + v^-$ with $v^{\pm} \in V^{\pm}$. We will realize $\omega_\psi^{(n,l)}$ on a "mixed" model. Let us denote $X = W \otimes V$ and $X^+ = W^+ \otimes V \oplus (e_0) \otimes V^-$ which will be viewed as $V^n \oplus V^-$. The space will be $\mathcal{S}(X^+)$; we view this space as the space of functions with n variables in V and one (the last) in V^-. We denote $Z_{n,n}$ the set of matrices

$$Z = \begin{pmatrix} z & & \\ & 1 & \\ & & z^* \end{pmatrix}$$

with $z \in \mathrm{N}_n$. On $\mathcal{S}(X^+)$, we have

$$\omega_\psi^{(n,l)}(\begin{pmatrix} z & & \\ & 1 & \\ & & z^* \end{pmatrix}, 1)\varphi(v_1, \dots, v_n; v^-) =$$

(1) $$\varphi(v_1, v_2 + z_{1,2}v_1, \dots, v_n + z_{1,n}v_1 + \dots + z_{n-1,n}v_{n-1}; v^-) \quad \text{with } z \in \mathrm{N}_n,$$

$$\omega_\psi^{(n,l)}(\begin{pmatrix} I_n & t & \\ & 1 & t' \\ & & I_n \end{pmatrix}, 1)\varphi(v_1, \dots, v_n; v^-) =$$

(2) $$\psi\Big(\Big(\sum_i t_i v_i^+, v^- + \sum_i t_i v_i^-\Big)\Big)\varphi\Big(v_1, \dots, v_n; v^- + \sum_i t_i v_i^-\Big),$$

(3) $$\omega_\psi^{(n,l)}(\begin{pmatrix} I_n & & S \\ & 1 & \\ & & I_n \end{pmatrix}, 1)\varphi(v_1, \dots, v_n; v^-)$$

$$= \psi(\mathrm{Tr}(\overline{\mathrm{Gram}(v)}.Sw_n))\varphi(v_1, \dots, v_n; v^-) \quad \text{with } \mathrm{Gram}(v) = ((v_i, v_j))_{1 \le i,j \le n},$$

(4) $$\omega_\psi^{(n,l)}(1, h)\varphi(v_1, \dots, v_n; v^-) =$$

$$|\det a|^{1/2}\psi(-(da^*v^-, v^-))\varphi(h^{-1}v_1, \dots, h^{-1}v_n; {a^*}^{-1}v^-) \quad \text{with } h = \begin{pmatrix} a & d \\ 0 & a^* \end{pmatrix}.$$

The $|\det a|^{1/2}$ in (4) is a normalizing factor to bring unitary representations to unitary representations; it corresponds to $|\det A|^{1/2}$ for a matrix $\begin{pmatrix} A & \\ & A^* \end{pmatrix}$ in the symplectic group of the space X.

Let ℓ be a Whittaker functional on τ. It satisfies

$$\ell(\tau(u).v) = \psi_X(u)\ell(v), \quad \forall u \in X(F_v), \forall v \in V_\tau.$$

We can view $\ell \circ T$ as a bilinear form b on $\mathcal{S}(X^+) \otimes V_\sigma$, satisfying

$$(5) \quad \forall(u,h,\xi) \in X(F_v) \times H_l \times V_\sigma, \quad b(\omega_\psi^{(n,l)}(u,h)\Phi, \sigma(h)\xi) = \psi_X(u)^{-1}b(\Phi,\xi).$$

We will show that the space of such bilinear forms is zero for $l < n$, and that if σ is non-zero for $l = n$, then it is generic. For that we will want to reduce the bilinear form to a smaller subspace. Since we have information on the action of a unipotent subgroup of G_n, we will study Jacquet modules of $\mathcal{S}(X^+)$. Let us first study the action of the $\mathrm{U}_{n,n}$ embedded in G_n in the outermost blocks. Using (3), we can study the twisted Jacquet module of $\mathcal{S}(X^+)$ with respect to the Siegel radical of this $\mathrm{U}_{n,n}$ and the trivial character. We see that if $\varphi(v_1,\dots,v_n;v^-) \neq 0$ then $\mathrm{Gram}(v) = 0$, so that the v_i generate a totally isotropic subspace of V; let us call it H. We still have the freedom to choose V^+, V^- and the f_i. We choose V^+ to be a maximal isotropic subspace of V containing H and $f_1,\dots, f_{\dim H}$ will be a basis of H. We then choose the f_i (and V^-) so that the matrix of $(.,.)$ in the basis $(f_1,\dots,f_n,f_{-n},\dots,f_{-1})$ is $\begin{pmatrix} & w_n \\ -w_n & \end{pmatrix}$. Now with (1) we can bring $(v_1,\dots,v_n)$ to an n-tuple $(v'_1,\dots,v'_n)$ with $v'_i = v_i$ if $v_i \notin \mathrm{Vect}_{1\leq j<i}(v_j)$ and $v'_i = 0$ otherwise. We want to mod-out by the action of the (upper) Siegel parabolic subgroup S_l of H_l. It can be described as the set of linear transformations h such that

- For any family (w_i), $\dim \mathrm{Vect}(w_i) = \dim \mathrm{Vect}(h(w_i))$ (invertible transform),
- For any w and w', $(w,w') = (h(w),h(w'))$ (unitary transform),
- For any family (w_i), $\dim \mathrm{Vect}(w_i^-) = \dim \mathrm{Vect}(h(w_i)^-)$ (in the Siegel parabolic subgroup).

Now the non-null vectors of (v'_i) form a free family of V^- and the scalar product of each pair is 0. Thus one can find in S_l an h such that $h(v'_i) = v''_i$ for any i provided that $v''_i = 0 \iff v'_i = 0$, $v''_i = f_k \iff v_i \notin \mathrm{Vect}_{1\leq j<i}(v_j)$ (note: k is a special variable that starts at 1 and increases by 1 each time it is used). Such families look like

$$(6) \qquad (0^*, f_1, 0^*, f_2, 0^*, \dots, 0^*, f_{\dim H}, 0^*);$$

0^* means "any number (including 0) of null vectors". Any two such families are not equivalent under the action of $Z_{n,n} \times S_l$. So we found a finite set of representatives for the action of $Z_{n,n} \times S_l$ on $W^+ \otimes V$.

We will now use equation (2). If a function φ in the twisted Jacquet module of $\mathcal{S}(X^+)$ is such that $\varphi(v) \neq 0$, then we must have

$$(7) \qquad \Big(\sum_i t_i v_i^+, v^- + \sum_i t_i v_i^-\Big) = \Big(\sum_i t_i v_i^+, v^-\Big) = t_n$$

for any such family of t_i. Since the product $(.,.)$ is non-degenerate, this gives $\dim H$ affine conditions on v^- with respect to the v_i. This constrains v^- to an affine subspace L° of V^- whose dimension is $l - \dim H$. The underlying vector space L is the orthogonal complement of H for the pairing defined by $(.,.)$ between V^+ and V^-. Thus a full set of representatives of the action of the product of the unipotent subgroup of G_n by S_l is given by

$$(8) \qquad (0^*, f_1, 0^*, f_2, 0^*, \ldots, 0^*, f_{\dim H}, 0^*; v_\circ^-)$$

where the first vectors verify equation (6), and $v_\circ^-$ is chosen to satisfy equation (7).

3.3. First Occurrence. We begin with

Proposition 3.1: *If $l < n$, the theta lift of π to H_l is trivial.*

Proof: The proof is purely local. We choose v a finite place of F as above. We want to prove that there is no such b as in equation (5). We have seen that if φ in the twisted Jacquet module of $\mathcal{S}(X^+)$ and $(v_1, \ldots, v_n; v^-)$ is such that $\varphi(v_1, \ldots, v_n; v^-) = 0$, then $(v_1, \ldots, v_n; v^-) \in X^\circ \simeq H^n \times L^\circ$. The set X° is a closed (affine) subspace of $X^+ \simeq V^n \times V^-$ and the set of elements of $\mathcal{S}(X^+)$ whose support is included in X° is isomorphic to $\mathcal{S}(X^\circ)$ (see [BZ, Proposition 1.8]). Now in X° we have several orbits of the subgroups considered in the preceding section. Each has a representative of the type (8). If we order the orbits by increasing dimension (which is the number of non-null vectors in the n first vectors of the representative), each orbit is closed in the union of the following ones; note that the order inside a given dimension class is not important. The twisted Jacquet module of $\mathcal{S}(X^+)$ for $X(F_v)$ and character

$$\psi_X \begin{pmatrix} z & t & S \\ & 1 & t' \\ & & z^* \end{pmatrix} = \psi\Big(\sum z_{i,i+1} + t_n\Big)$$

is composed of functions with support included in the union of these orbits. The set of such functions on the orbit of a vector of type (8) is isomorphic to $\mathrm{Ind}^{\mathrm{c}}{}_R^{Z_{n,n} \times S_l}\, \mathcal{S}(L^\circ)$, where $\mathrm{Ind}^{\mathrm{c}}$ is compact induction and $R(F_v)$ is the stabilizer of (8) in $Z_{n,n} \times S_l$ (the action of $Z_{n,n}$ is trivial on $\mathcal{S}(L^\circ)$ and that of S_l is the standard one). Thus the bilinear form b would be a $Z_{n,n} \times S_l$-invariant bilinear form on

$$\mathrm{Ind}^{c\,Z_{n,n}\times S_l}_{R}\ \mathcal{S}(L^\circ) \times (\psi_X \otimes \sigma).$$

But now if $\dim H < n$, $R(F_v)$ contains a subgroup of the form $J \times \{1\}$ where J is a simple root subgroup of G_n and on that subgroup ψ_X is non-trivial, thus b is zero. ∎

This gives

COROLLARY 3.1:

- *If π lifts non-trivially to $H_n(\mathbf{A}_F)$ then the lift is cuspidal and generic.*
- *If τ is a generic representation of $G_n(F_v)$ that lifts non-trivially to a representation σ of $H_n(F_v)$ then σ is generic.*

Proof: Combining Proposition 3.1 with the second part of [Wat1, Theorem 4.3, p. 251], we get that the lift is cuspidal.

We prove the local result concerning Whittaker models. Suppose that τ is a representation of $G_n = G_n(F_v)$ with Whittaker model with respect to character ψ_X. Suppose that it lifts to a non-trivial representation σ of $H_n(F_v)$. This means that the bilinear form b is non-trivial. But the space of such bilinear forms is isomorphic to

$$(9)\quad \mathrm{Hom}_{Z_{n,n}\times S_n}(\psi_X \otimes \sigma, \mathrm{Ind}^{c\,Z_{n,n}\times S_n}_{R}\ \mathcal{S}(L^\circ)) \simeq \mathrm{Hom}_R(\mathrm{Res}_R(\psi_X \otimes \sigma), \mathcal{S}(L^\circ)).$$

We must have $\dim H = n$ so that there are no null vectors in the first n elements of (8). We thus have only one possibility for the last element: f_{-n}. Thus $L^\circ = \{f_{-n}\}$ and $\mathcal{S}(L^\circ)$ is the trivial representation. The representative for the class of $(v_1, \ldots, v_n; v^-)$ can then be chosen equal to

$$(10)\qquad (f_1, f_2, \ldots, f_n; f_{-n}).$$

Then we have

$$R(F_v) = \left\{ \left(\begin{pmatrix} z & & \\ & 1 & \\ & & z^* \end{pmatrix}, \begin{pmatrix} z & * \\ & z^* \end{pmatrix} \right) \right\}.$$

The homomorphism of (9) is a function ℓ on V_σ such that for any $u = \begin{pmatrix} z & * \\ & z^* \end{pmatrix}$ and any $\xi \in V_\sigma$,

$$\ell(\psi(z_{1,2} + \cdots + z_{n-1,n})\sigma(u)(\xi)) = \ell(\xi),$$

or equivalently

$$\ell(\sigma(u)\xi) = \psi(z_{1,2} + \cdots + z_{n-1,n})^{-1}\ell(\xi).$$

So this ℓ is a Whittaker functional on V_σ. This proves that the lift is generic. ∎

3.4. From $U_{n+1,n}$ to $U_{n,n}$. The embedding of a pair of unitary groups in a metaplectic group depends on a character. We will call this character the parameter of the corresponding Weil representation and Θ correspondence.

Proposition 3.3: *Assume $P_\psi(\varphi, \mu)$ is not identically 0 as φ varies in the space of π. Then the Θ lift of $\pi \otimes \mu \circ \det$ to $U_{n,n}$ with respect to character ψ and some parameter ν is non-trivial. As noted before, this means that it is cuspidal and generic.*

Proof: The proof is very similar to the proof of Proposition 3.1 and Corollary 3.1. The Weil representation we choose now will be roughly the opposite one. This time we pick $X^+ = W \otimes V^+$ (identified with W^n); G_n still acts on the right and H_n still on the left. As in [Kud, Proposition 3.1], we choose a character ν of $\mathbf{A}_E^\times$ whose restriction to $\mathbf{A}_F^\times$ is the quadratic character corresponding to the extension E/F. For convenience, we twist the action of G_n by the character $\mu \circ \det$. The action of the subgroups is comparatively easy to describe. We have

(11) $$\omega_{\psi,\nu}(g, 1)\Phi(x_1, \ldots, x_n) = \mu(\det g)\Phi(x_1 g, \ldots, x_n g),$$

(12) $$\omega_{\psi,\nu}(1, \begin{pmatrix} a & \\ & a^* \end{pmatrix})\Phi(x_1, \ldots, x_n) = \nu(\det a^*)^{2n+1}|\det a|^{n+1/2}\Phi(a^{-1} \cdot (x_1, \ldots, x_n)),$$

(13) $$\omega_{\psi,\nu}(1, \begin{pmatrix} I_n & S \\ & I_n \end{pmatrix})\Phi(x_1, \ldots, x_n) = \psi\Big(\frac{1}{2}\operatorname{Tr}(\overline{\operatorname{Gram}(x)}.Sw_n)\Big)\Phi(x_1, \ldots, x_n),$$

Remark: $a \cdot (x_1, \ldots, x_n) = (\sum a_{ij}x_j)_{1 \le i \le n}$ (this is the formal action of a on a column vector, except that instead of being scalars, the x_i are vectors); the term $|\det a|^{n+1/2}$ in (12) is the normalizing factor.

We denote the Θ lift of π with respect to Weil representation $\omega_{\psi,\nu}$ by $\theta_{\psi,\nu}(\pi)$. Notice that this is the Θ lift of $\pi \otimes \mu \circ \det$. Then the elements of $\theta_{\psi,\nu}(\pi)$ are the functions

(14) $$\xi(h) = \int_{G_n(F)\backslash G_n(\mathbf{A}_F)} \theta^\Phi_{\psi,\nu}(g, h)\varphi(g)\mathrm{d}g, \quad \varphi \in \pi, \quad h \in U_{n,n}(\mathbf{A}_F),$$

where $\theta^{\Phi}_{\psi,\nu}$ is the Θ kernel for the dual pair (G_n, H_n) with Weil representation $\omega_{\psi,\nu}$. It is defined by

$$\theta^{\Phi}_{\psi,\nu}(g,h) = \sum_{x\in X^+(F)} \omega_{\psi,\nu}(g,h)\Phi(x),$$

where Φ is a Schwartz–Bruhat function on X^+.

To prove the non-vanishing of $\theta_{\psi,\nu}(\pi)$, we will directly compute its Whittaker coefficient with respect to the upper triangular unipotent subgroup U_H of H_n. We want to compute

$$W_\xi(h) = \int_{U_H(F)\backslash U_H(\mathbf{A})} \xi(vh)\psi(v)^{-1}\mathrm{d}v \tag{15}$$

with ξ as above. We substitute ξ with the expression in (14), substitute the θ expression and perform the integration with respect to the Siegel radical of H_n. What remains of the sum over $X^+(F)$ are the vectors $x = (x_1,\dots,x_n)$ such that

$$\mathrm{Gram}(x) = (\langle x_i, x_j\rangle) = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 1 \end{pmatrix}, \tag{16}$$

that is, all the products are 0 except $\langle x_n, x_n\rangle$, which is 1. If we take

$$Z' = \left\{ \begin{pmatrix} z & \\ & z^* \end{pmatrix} \in \mathrm{U}_{n,n} \| z \in \mathrm{N}_n \right\},$$

we have

$$W_\xi(h) = \int_{Z'(F)\backslash Z'(\mathbf{A}_F)} \int_{G_n(F)\backslash G_n(\mathbf{A}_F)} \sum \omega_{\psi,\nu}(g, zh)\Phi(x_1,\dots,x_n)\varphi(g)\mathrm{d}g\psi(z)^{-1}\mathrm{d}z,$$

the sum being over all $x = (x_1,\dots,x_n) \in X^+$ satisfying (16). If the vectors $(x_1,\dots,x_{n-1})$ are not linearly independent, as in the proof of Proposition 3.1, there is a simple root subgroup in their stabilizer and the intertwining operator vanishes as well as the integral. So we have only one orbit under the action of $G_n(F)$ and we can take as a representative of x in its orbit the system $(e_1,\dots,e_{n-1},e_0)$. Its stabilizer in G_n is the subgroup

$$R' = \left\{ r = \begin{pmatrix} I_{n-1} & * & 0 & * & * \\ & a & 0 & b & * \\ & & 1 & 0 & 0 \\ & c & & d & * \\ & & & & I_{n-1} \end{pmatrix} \middle\| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{U}_2 \right\}.$$

This means

$$W_\xi(h) = \int_{Z'(F)\backslash Z'(\mathbf{A}_F)} \int_{R'(\mathbf{A}_F)\backslash G_n(\mathbf{A}_F)} \omega_{\psi,\nu}(g, zh)\Phi(e_1,\ldots,e_{n-1},e_0)\varphi^{R'}(g)\mathrm{d}g\psi(z)^{-1}\mathrm{d}z,$$

where $\varphi^{R'}(g) = \int_{R'(F)\backslash R'(\mathbf{A}_F)} \varphi(rg)\mu(\det r)\mathrm{d}r$. Now because the representative is so specific, we can use the Weil representation to transform the integral over Z' in the integral over some subgroup of H_n. We have

$$\omega(g, \begin{pmatrix} z & & & \\ & 1 & & \\ & & 1 & \\ & & & z^* \end{pmatrix} h)\Phi(e_1,\ldots,e_{n-1},e_0) = \omega(\begin{pmatrix} z & & \\ & I_3 & \\ & & z^* \end{pmatrix}^{-1} g, h)\Phi(e_1,\ldots,e_{n-1},e_0), \tag{17}$$

$$\omega(g, \begin{pmatrix} I_{n-1} & \begin{matrix} v_1 \\ \vdots \\ v_{n-1} \end{matrix} & & \\ & 1 & & \\ & & 1 \quad -\overline{v_{n-1}} \quad \cdots \quad -\overline{v_1} & \\ & & & I_{n-1} \end{pmatrix} h)\Phi(e_1,\ldots,e_{n-1},e_0) =$$

$$\omega(\begin{pmatrix} I_n & \begin{matrix} v_1 \\ \vdots \\ v_{n-1} \\ 0 \end{matrix} & * \\ & 1 & 0 \quad -\overline{v_{n-1}} \quad \cdots \quad -\overline{v_1} \\ & & I_n \end{pmatrix}^{-1} g, h)\Phi(e_1,\ldots,e_{n-1},e_0) \tag{18}$$

with $*$ such that the resulting matrix is in G_n. We can then exchange the order of integration and what remains is

$$\begin{aligned} W_\xi(h) = &\int_{R'(\mathbf{A}_F)\backslash G_n(\mathbf{A}_F)} \omega_{\psi,\nu}(g,h)\Phi(e_1,\ldots,e_{n-1},e_0) \\ &\qquad \int_{Z''(F)\backslash Z''(\mathbf{A}_F)} \varphi^{R'}(zg)\psi(z)^{-1}\mathrm{d}z\mathrm{d}g \\ = &\int_{R'(\mathbf{A}_F)\backslash G_n(\mathbf{A}_F)} \mu(\det g)\omega_{\psi,\nu}(1,h)\Phi(e_1 g,\ldots,e_{n-1}g,e_0 g) \\ &\qquad \int_{Z''(F)\backslash Z''(\mathbf{A}_F)} \varphi^{R'}(zg)\psi(z)^{-1}\mathrm{d}z\mathrm{d}g, \end{aligned}$$

where Z'' is the subgroup of G_n deduced from Z' thanks to equations (17) and (18). We see that the innermost integral with $g = I_{2n+1}$ is just $P_\psi(\varphi,\mu)$. This means that it is non-zero for some φ by hypothesis. Since Φ is arbitrary, we can choose it such that the support of the function

$$g \mapsto \omega_{\psi,\nu}(1,h)\Phi(e_1 g,\ldots,e_{n-1}g,e_0 g)$$

is concentrated as near I_n as we want and for a Φ with a small enough support; W_ξ will be non-zero at that h, thus the Θ lift is both non-zero and generic. ∎

Note that, as already proved in [Wat1, Theorem 4.6], the lift to H_{n+1} is always non-trivial and generic. If we repeat the proof of the last proposition, we will find

$$W_\xi(h) = \int_{U_0(\mathbf{A}_F)\backslash G_n(\mathbf{A}_F)} \omega_{\psi,\nu}(g,h)\Phi(e_1,\ldots,e_{n-1},e_n,e_0)W_\varphi(g)\mathrm{d}g.$$

Since Φ is arbitrary, it can be chosen so that the integral is non-zero.

4. Existence of the pole

In this section we prove that if $\pi\otimes\mu\circ\det$ comes from a representation σ with respect to some Θ lifting, then $\mathrm{L}^S(\pi\times\mu,s)$ has a (necessarily simple) pole at $s=1$ for any finite set of places $S\supset S_0$. Let us suppose that $\pi\otimes\mu\circ\det$ comes through Θ correspondence from a representation σ of H_n and let ν be the character determining the splitting. We suppose that the action of G_n on the Schrödinger space is, up to the unitary normalizing factor, linear.

We know from section 3 that σ is necessarily generic. We first recall from [Wat2, p. 113] the relation between the values of the Whittaker functions. Let

φ be an element of σ. For each $\Phi \in \mathcal{S}(X^+)$, we denote φ_Φ the element of $\pi \otimes \mu \circ \det$ defined by the formula

$$\varphi_\Phi(g) = \int_{H_n(F)\backslash H_n(\mathbf{A}_F)} \theta^\Phi_{\psi,\nu}(g,h)\varphi(h)\mathrm{d}h.$$

We then have

$$W_{\varphi_\Phi}(g) = \int_{U_H(\mathbf{A}_F)\backslash H_n(\mathbf{A}_F)} W_\varphi(h) \int_{Z_{n,n}(\mathbf{A}_F)} \psi(z)^{-1}\omega(zg,h)\Phi(f_1,\dots,f_n;f_{-n})\mathrm{d}z\mathrm{d}h.$$

Remark: Here $U_H(\mathbf{A}_F)\backslash H_n(\mathbf{A}_F)$ is not a group, so that $\mathrm{d}h$ is not a Haar measure. By the Bruhat decomposition, an element h can be written $U_H(\mathbf{A}_F)ak$ with a in the split torus and k in the maximal compact subgroup of H_n. We then have $\mathrm{d}h = \delta^{-1}(a)\mathrm{d}a\mathrm{d}k$, because if $g = nak$, $\mathrm{d}g = \delta^{-1}(a)\mathrm{d}n\mathrm{d}a\mathrm{d}k$.

This formula decomposes into an Euler product, so we will use it locally. We suppose that v is a place of F outside S_0. We take the model of the Θ correspondence that we had in the proof of Proposition 3.1. Suppose that E/F remains inert at v and let w be the place of E above v. We denote ϖ the uniformizer of $E_w = E \otimes_F F_v$ and q the number of elements of the residual field of F_v. We choose Φ to be the characteristic function of $X^+(\mathcal{O}_v)$. We notice that W_{φ_Φ} is right invariant by the action of $G_n(\mathcal{O}_v)$, so that we found the essential vector, provided it is non-zero. We take

$$g = \begin{pmatrix} \varpi^{a_1} & & & & \\ & \ddots & & & \\ & & \varpi^{a_n} & & \\ & & & 1 & \\ & & & & d^* \end{pmatrix}$$

with a_i integers and d^* the appropriate diagonal matrix. Using the above formula, we find

$$W_{\varphi_\Phi}(g) = \sum_{b_1 \geq \dots \geq b_n} \delta^{-1}(h)W_\varphi(h) \int_{Z_{n,n}(\mathbf{A}_F)} \psi(z)^{-1}\omega(zg,h)\Phi(f_1,\dots,f_n;f_{-n})\mathrm{d}z$$

with

$$h = \begin{pmatrix} \varpi^{b_1} & & & \\ & \ddots & & \\ & & \varpi^{b_n} & \\ & & & d'^* \end{pmatrix}.$$

We have

$$\omega(zg,h)\Phi(f_1,\dots,f_n;f_{-n}) =$$
$$q^{-n\sum a_i-\frac{1}{2}\sum b_i}\nu_v(\prod(\varpi^{b_i})^*)\Phi(\varpi^{a_1-b_1}f_1,\varpi^{a_2-b_2}f_2+\varpi^{a_2-b_1}z_{12}f_1,\dots;\overline{\varpi^{b_n}}f_{-n})$$

with $z=(z_{ij})$. The power of q on the first line is the normalization factor to bring unitary representations to unitary representations. To contribute non-trivially, we first see that we must take $a_1\geq b_1$. Then we need to have $\varpi^{a_2-b_1}z_{12}\in\mathcal{O}_v$ for the non-vanishing of Φ, so $z_{12}\in\varpi^{b_1-a_2}\mathcal{O}_v$. If $a_2>b_1$, since ψ_v is trivial exactly on $\mathcal{O}_v$, the integral over z_{12} will be equal to 0. We thus must take $b_1\geq a_2$. We see that we have to take $a_2\geq b_2$. Iterating yields

$$a_1\geq b_1\geq a_2\geq\cdots\geq a_n\geq b_n\geq 0.$$

The variables z_{ij}, $i<j$, vary in $\varpi^{b_i-a_j}\mathcal{O}_v$. This yields

$$W_{\varphi_\Phi}(g)=\sum_{a_1\geq b_1\geq\cdots\geq a_n\geq b_n\geq 0}\bar\nu(\varpi^{-\sum b_i})W_\varphi(h)q^{\sum(-n+i-1)a_i+(n-i+\frac{1}{2})b_i}$$

with $\bar\nu(x)=\nu(\bar x)$. We can thus compute the local L-function of $\pi_v\times\mu_w$ in terms of σ_v:

$$\begin{aligned}\mathrm{L}(\pi_v\times\mu_w,s)&=\sum_{a_1\geq 0}W_{\varphi_\Phi}(g)q^{-a_1(s-n)}\quad\text{all other } a_i=0\\&=\sum_{a_1\geq b_1\geq 0}\bar\nu(\varpi)^{-b_1}W_\varphi(h)q^{-na_1+(n-1/2)b_1-a_1(s-n)}\\&=\sum_{a_1\geq b_1\geq 0}\bar\nu(\varpi)^{-b_1}W_\varphi(h)q^{-b_1(s-n+1/2)+(b_1-a_1)s}\\&=\sum_{m\geq 0}q^{-ms}\sum_{b_1\geq 0}\bar\nu(\varpi)^{-b_1}W_\varphi(h)q^{-b_1(s-n+1/2)}\\&=\zeta_{E,w}(s)\mathrm{L}(\sigma_v\times\bar\nu_v^{-1},s).\end{aligned}$$

The fact that the sum on the next to last line is equal to $\mathrm{L}(\sigma_v\times\bar\nu_v^{-1},s)$ will be proven as part of an upcoming paper on the reverse case; it is anyway similar to the formula found for $\mathrm{L}(\pi_v\times\mu_w,s)$ in section 2.

This proof can be conducted along the same lines for v split (but we get a second variable instead of the conjugate one) for the same result.

The bottom line is that for a finite set of places S containing S_0, we have

$$\mathrm{L}^S(\pi\times\mu,s)=\zeta_E^S(s)\mathrm{L}^S(\sigma\times\bar\nu^{-1},s).$$

Since σ is unitary and generic, $\mathrm{L}^S(\sigma\times\bar\nu^{-1},s)$ cannot vanish at $s=1$ so that the partial L-function of $\pi\times\mu$ must have a pole there.

We recall that on the right-hand side, the character μ of the beginning is built into σ by the twisting of the Weil representation. If we put $\pi' = \pi \otimes \mu$ and twist the representation by $\mu \circ \det^{-1}$ (on the G_n side), what we get is

$$\mathrm{L}^S(\pi' \times \mu, s) = \mathrm{L}^S(\bar{\mu}, s)\mathrm{L}^S(\sigma \times \bar{\mu}\bar{\nu}^{-1}, s).$$

This is the analog of the result of T. Watanabe, [Wat2, p. 94].

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